THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Supplementary Exercise 1

1. Define a relation $\sim$ on $\mathbb{R}^2$ such that $(x, y) \sim (x', y')$ if and only if $x - x', y - y' \in \mathbb{Z}$.
   (a) Prove that $\sim$ is an equivalence relation.
   (b) Describe the elements of $\mathbb{R}^2/\sim$.
   (c) Repeat (a) and (b) by changing the relation to be the following:
   $(x, y) \sim (x', y')$ if and only if $x - x' \in \mathbb{Z}$ and $y = y'$.
2. Let $M_n(\mathbb{R})$ be the set of all n by n real matrices. Suppose that $\sim$ is a relation on $M_n(\mathbb{R})$ defined by $A \sim B$ if there exists an invertible matrix Q such that $B = AQ$.
   (a) Prove that $\sim$ is an equivalence relation.
   (b) Describe the elements of the equivalence class which contains the identity matrix I.
3. Let n be a positive integer and let $\sim$ be a relation defined on $\mathbb{Z}$ which is given by $a \sim b$ if $b - a$ is divisible by n.
   (a) Show that $\sim$ is an equivalence relation.
   (b) Write down the elements of $\mathbb{Z}_n := \mathbb{Z}/\sim$.
   (c) Prove that multiplication on $\mathbb{Z}$ induces a multiplication on $\mathbb{Z}_n$.
   (d) What is the remainder when 7001×492 is divided by 7?
   (Hint: What is $[7001 \cdot 492]$ in $\mathbb{Z}_7$?)
4. For an incidence geometry, prove that two distinct lines can have most one point in common, i.e. if l and m are distinct lines, then $|l \cap m| \leq 1$.
5. For an incidence geometry, prove that:
   (a) if P be a point, then there exists at least one line that does not contain P;
   (b) there exist three distinct lines such that no point lies on all three of them.

Lecturer's comment:

1. (a) (i) Let $(x, y) \in \mathbb{R}^2$, since $x - x = y - y = 0 \in \mathbb{Z}$, so $(x, y) \sim (x, y)$
   (ii) Let $(x, y), (x', y') \in \mathbb{R}^2$ and $(x, y) \sim (x', y')$. Then $x - x', y - y' \in \mathbb{Z}$, which implies that $x' - x = -(x - x')$ and $y' - y = -(y - y')$ are in $\mathbb{Z}$ and so $(x', y') \sim (x, y)$.
   (iii) Let $(x, y), (x', y'), (x'', y'') \in \mathbb{R}^2$ such that $(x, y) \sim (x', y')$ and $(x', y') \sim (x'', y'')$. Then $x - x', x' - x'', y - y', y' - y'' \in \mathbb{Z}$. Therefore, $x - x'' = (x - x') + (x' - x'') \in \mathbb{Z}$ and $y - y'' = (y - y') + (y' - y'') \in \mathbb{Z}$. Hence, $(x, y) \sim (x'', y'')$.

   Therefore, $\sim$ is an equivalence relation on $\mathbb{R}^2$.

(b) $\mathbb{R}^2/\sim = \{[(x,y)] : 0 \le x, y < 1\}$.

(Remark: if you regard $\mathbb{R}^2$ as a piece of paper and try to glue the points which are related by $\sim$, then you will get a torus.)

(c) The proof is similar to (a) and $\mathbb{R}^2/\sim = \{[(x,y)] : 0 \le x < 1,\, y \in \mathbb{R}\}$. Again $\mathbb{R}^2/\sim$ may be regarded as a cylinder.

2. (a) (i) Let $A \in M_n(\mathbb{R})$, since $A = AI$ where I is the identity matrix which is invertible, $A \sim A$.

(ii) Let $A, B \in M_n(\mathbb{R})$ and $A \sim B$, then there exists an invertible matrix Q such that $B = AQ$. Then, we have $A = BQ^{-1}$ where Q^{-1} is an invertible matrix and so $B \sim A$.

(iii) Let $A, B, C \in M_n(\mathbb{R})$ such that $A \sim B$ and $B \sim C$. Then there exist invertible matrices P and Q such that $A = BP$ and $B = CQ$. Therefore, $A = (CQ)P = C(PQ)$. Note that the product of two invertible matrices is an invertible matrix, so PQ is invertible and $A \sim C$.

Therefore, $\sim$ is an equivalence relation on $M_n(\mathbb{R})$.

(b) Note that $[I] = \{P \in M_n(\mathbb{R}) : P \sim I\}$.

We claim that $[I]$ is the set of all invertible matrices, which is denoted by $GL_n(\mathbb{R})$.

Firstly, if $P \in [I]$, then $P \sim I$ which means $P = IQ = Q$ for some invertible matrix Q. Therefore, P is invertible and $[I] \subset GL_n(\mathbb{R})$.

Secondly, if $P \in GL_n(\mathbb{R})$, i.e. P is invertible. If we want to show $P \in [I]$, we have to show that $P \sim I$, i.e. there exists some invertible matirx Q such that $P = IQ$, but it is true simply by taking $Q = P$. Therefore, $GL_n(\mathbb{R}) \subset [I]$.

Therefore, $[I] = GL_n(\mathbb{R})$.

(Remark: To show two sets A and B are the same, a standard way is showing that both $A \subset B$ and $B \subset A$ are true.)

3. (a) Let a, b and c be integers.

Since $a - a = 0$ which is divisible by n, $a \sim a$.

Suppose that $a \sim b$, then $b - a = np$ for some integer p.

Then $a - b = -np = n(-p)$ which is divisible by n, so $b \sim a$.

Suppose that $a \sim b$ and $b \sim c$, then $b - a = np$ and $c - b = nq$ for some integers p and q.

Then $c - a = (c - b) + (b - a) = n(p + q)$. $p + q$ is an integer, so $c - a$ is divisible by n and $c \sim a$.

As a result, $\sim$ is an equivalence relation.

(b) $\mathbb{Z}_n := \mathbb{Z}/\sim = \{[0], [1], \cdots, [n]\}$.

(c) It suffices to show that if $a \sim a'$ and $b \sim b'$ then $a \cdot b \sim a' \cdot b'$.

Suppose that $a' - a = np$ and $b' - b = nq$ for some integers p and q.

Then $(a' \cdot b') - (a \cdot b) = (a + np) \cdot (b + nq) - a \cdot b = n(aq + bp + npg)$. $aq + bp + npg$ is an integer, so $(a' \cdot b') - (a \cdot b)$ is divisible by n and $a \cdot b \sim a' \cdot b'$.

(d) Note that $[7001] = [1]$ and $[492] = [2]$ in $\mathbb{Z}_7$, so $[7001 \cdot 492] = [7001] \cdot [492] = [1] \cdot [2] = [2]$.

Therefore, when 7001×492 is divided by 7, the remainder is 2.

4. Suppose that p and q are two mathematical statementg. If we want to show that the statement $p \to q$ is true, here are two of the ways to do:

   - (Prove by contrapositive) Prove that $(\neg q) \to (\neg p)$, which is logically equivalent to $p \to q$, is true.
   - (Prove by contradiction) We want to show the negation of the statement we want to prove is false, i.e. contradiction exists. Note that $p \to q$ is logically equivalent to $\neg p \vee q$ and so its negation is $p \wedge (\neg q)$.

   For the statement in the question, p is the statement "l and m are distinct lines", q is the statement $|l \cap m| \leq 1$.

   We will show $p \to q$ is true by using different methods:

   (Prove by contrapositive) Suppose that $|l \cap m| > 1$ $(\neg q)$, i.e. there exist two points A and B such that both A and B lie on l as well as m. By axiom **I1**, l and m must be the same $(\neg p)$.

   (Prove by contradiction) Suppose that l and m are distinct lines and $|l \cap m| > 1$ $(p \wedge (\neg q))$. Then there exist two points A and B such that both A and B lie on l as well as m. By axiom **I1**, l and m must be the same which is a contradiction.

5. (a) By axiom **I3**, there exist three noncollinear points R, S and T.

   (Case 1) $P \in \{R, S, T\}$

   Without loss of generality, let $R = P$.
   By axiom **I1**, there exists unique line l_{ST} such that $S, T \in l_{ST}$.
   Note that l_{ST} does not contain P, otherwise it contradicts to the assumption that P, S and T are noncollinear.

   (Case 2) $P \notin \{R, S, T\}$

   By axiom **I1**, there exists unique lines l_{ST} such that $S, T \in l_{ST}$.
   If P does not lie on l_{ST}, then l_{ST} is the line required.
   If $P \in l_{ST}$. By axiom **I1**, there exists unique line l_{RS} such that $R, S \in l_{RS}$.
   If P lies on l_{RS}, then both P and S lie on l_{ST} and l_{RS}. By axiom **I1**, $l_{ST} = l_{RS}$ which is a line that contains R, S and T (Contradiction).
   Therefore, P does not lie on l_{ST}

   (b) By axiom **I3**, there exist three noncollinear points R, S and T.

   By axiom **I1**, there exist unique line l_{RS}, l_{ST} and l_{RT} such that $R, S \in l_{RS}$, $S, T \in l_{ST}$ and $R, T \in l_{RT}$.

   Firstly, l_{RS}, l_{ST} and l_{RT} are distinct lines, otherwise two of them will be the same line which contains all R, S and T which is a contradiction.

   Secondly, if there exists a point P such that P lies on all three of them, in particular P lies on l_{RS} and l_{ST} which forces $P = S$ (By question 4 or you may say it is a direct consequence of axiom **I1**. However, $P = S$ which lies on l_{RT} which contradicts to the assumption that P, S and T are noncollinear.

   Therefore, there exists no point which lies on both l_{RS}, l_{ST} and l_{RT}.